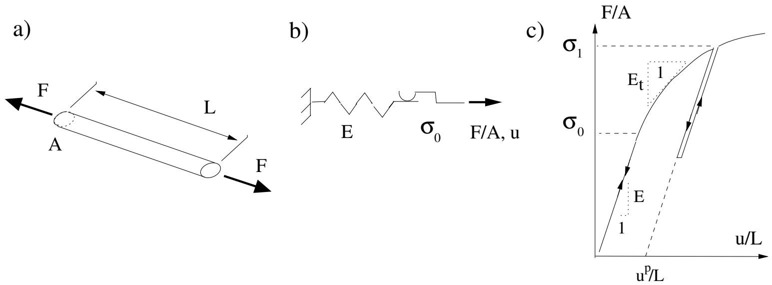
Chapter III, Non-linear material response
Poro-plasticity theory

Lecture no 6

9 April 2009

Part I : 1D plasticity theory

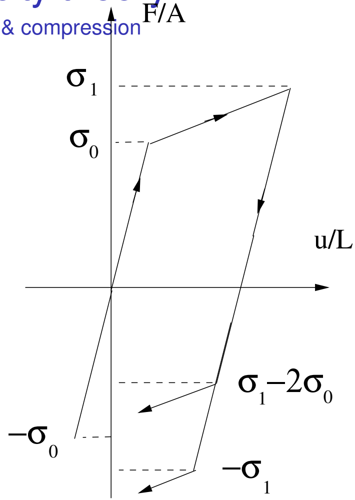
The extension of a bar



- The slider is rigid for $\sigma \leq \sigma_0$. It slips for larger stress but requires increasing the load to keep on doing so : work-hardening.
- Upon unloading the trajectory is parallel to the initial elastic response.
- The material keeps memory of the last stress necessary to activate the slider.

Part I : 1D plasticity theory

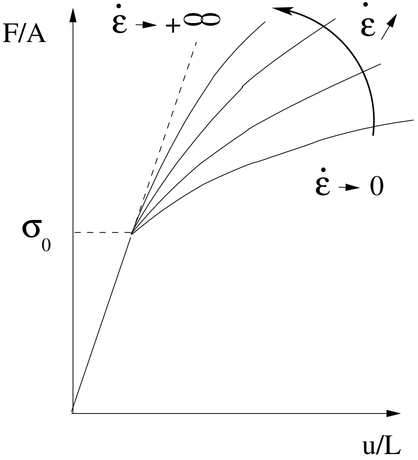
A cycle of loading : extension & compression



- Upon unloading, the limit of elasticity is felt at $\sigma_1 - 2\sigma_0$ (kinematic hardening) or at $-\sigma_1$ (isotropic hardening).
- The reality is anything in between, Bauschinger effect.

Part I : 1D plasticity theory

Strain-rate dependency, visco-plasticity



- Two time scales : the loading and the motion of the defects (typically, dislocation for crystalline materials).

Part I : 1D plasticity theory

The 1D theory

• Objective : the construction of a mathematical theory to represent the experimental results.

• Require a family of hidden or internal variables to describe the evolution of the micro-structure.

Could also receive a phenomenological description.

Consider a single variable here, γ^p, the cumulated plastic deformation, for isotropic hardening.

• Consider first a time-independent material response : Time is defined by the loading.

• $\dot{\sigma}$ is called the stress rate or the stress increment.

Part I : 1D plasticity theory

- Initial elastic domain : $[-\sigma_0; +\sigma_0]$

Suppose σ_0 function of γ^p for isotropic hardening.

$$\phi(\sigma, \gamma^p) \equiv |\sigma| - \sigma_0(\gamma^p) < 0\,. \tag{1}$$

The stress-strain relation for elasticity : $\dot{\sigma} = E\dot{\epsilon}$

- First plasticity : reach the domain boundary, $\phi(\sigma, \gamma^p) = 0$.

This condition is not enough since two cases are to be considered.
For tensile test :

if $\dot{\sigma}$ positive, plastic flow.

if $\dot{\sigma}$ negative, elastic response.

- Introduce the external normal to the elastic domain :

$N = \sigma/|\sigma|$ is the sign of the stress

- Plasticity in tension or compression requires

$$\phi(\sigma, \gamma^p) = 0\,, \quad N\dot{\sigma} > 0\,. \tag{2}$$

- Small perturbation assumption, additive decomposition of the strain rate

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p \,, \tag{3}$$

- Elastic part still linked to stress via Hooke's law

$$\dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}^p) \,. \tag{4}$$

- The rate of plastic deformation defined by the flow rule :

$$\dot{\epsilon}^p = \dot{\gamma}^p N \,, \tag{5}$$

$\dot{\gamma}^p$, the rate of accumulated plastic deformation, is a positive scalar.

Direction of flow given by the normal but what is the intensity ?

• Consistency condition : the stress during flow remains on the elastic domain boundary

$$\phi = 0 \quad \text{and} \quad \dot{\phi} = 0 \quad \text{during} \quad \text{plastic} \quad \text{flow.}$$

Consequently :

$$\dot{\phi}(\sigma, \gamma^p) \equiv \frac{\partial \phi}{\partial \sigma}\dot{\sigma} + \frac{\partial \phi}{\partial \gamma^p}\dot{\gamma}^p = N\dot{\sigma} - \sigma_0'\dot{\gamma}^p = 0 \ , \tag{6}$$

where σ_0' (noted also h) is the first derivative of σ_0 wrt γ^p.

$$\dot{\gamma}^p = \frac{N}{h}\dot{\sigma} \ . \tag{7}$$

• Express directly $\dot{\gamma}^p$ in terms of the strain rate :

Replace $\dot{\sigma}$ with Hooke's law (4)

$$\dot{\gamma}^p = \frac{N}{h} E(\dot{\epsilon} - \dot{\epsilon}^p) \ .$$

and use plastic flow rule $\dot{\epsilon}^p = \dot{\gamma}^p N$ to obtain

$$\dot{\gamma}^p = \frac{NE}{h+E} \dot{\epsilon} \ . \tag{8}$$

• The tangent modulus E_t :
Combine with Hooke's law (4) to obtain

$$\dot{\sigma} = E_t \dot{\epsilon} \, , \quad E_t = \frac{Eh}{h+E} \ . \tag{9}$$

- Validity of this construction : h has to be different from zero.

Otherwise : the rate $\dot{\gamma}^p$ requires the solution of the whole boundary value problem.

This is called perfect-plasticity.

• Same definition of the elastic domain (1).

• The stress points does not stay on the boundary during flow but enters the domain $\phi > 0$, forbidden in rate-independent plasticity.

The size of the elastic domain $\sigma_0(\gamma^p)$ smaller than $|\sigma|$.

• The rate of plastic flow defined by ode.

Simple case of linear over-stress :

$$\dot{\gamma}^p = \frac{|\sigma| - \sigma_0(\gamma^p)}{\eta} \, , \tag{10}$$

η, the viscosity (*Pas*)

$\dot{\gamma}^p = 0$ if condition $|\sigma| \geq \sigma_0(\gamma^p)$ not met.

Flow rule remains unchanged : $\dot{\epsilon}^p = \dot{\gamma}^p N$

The distance $|\sigma| - \sigma_0(\gamma^p)$ defines the intensity of the rate of plastic deformation.

Part II :

Multi-axial plasticity theory

- The normal to the yield surface
- The flow potential
- The elasto-plastic tangent

Part II : Multi-axial plasticity theory

The elasticity domain

- The elastic domain is defined in the stress space (dim 6).

$$\phi(\underline{\underline{\sigma}}, \gamma^p) \leq 0$$

with single internal variable γ^p for isotropic hardening.

- Convexity of the domain :

$$\phi(\underline{\underline{\sigma}}^{(a)}, \gamma^p) \leq 0 \qquad \phi(\underline{\underline{\sigma}}^{(b)}, \gamma^p) \leq 0$$

then

$$\phi(\underline{\underline{\sigma}}(a)\alpha + \underline{\underline{\sigma}}^{(b)}(1-\alpha), \gamma^p) \leq 0 \quad \forall \alpha \in [0;1]\,.$$

Part II : Multi-axial plasticity theory

The normal to the elastic domain

- The infinitesimal variation in $d\underline{\underline{\sigma}}$ leads to $d\phi$.

The two are related by :

$$d\phi = \nabla_\sigma \phi : d\underline{\underline{\sigma}} \quad \left(d\phi = \frac{\partial \phi}{\partial \sigma_{ij}} d\sigma_{ji} \right), \tag{11}$$

where $\nabla_\sigma \phi$, the gradient of a scalar wrt to stress, is 2nd order tensor.

If variation $d\underline{\underline{\sigma}}$ permits to stay on the boundary $\phi = 0$ then $d\phi = 0$.

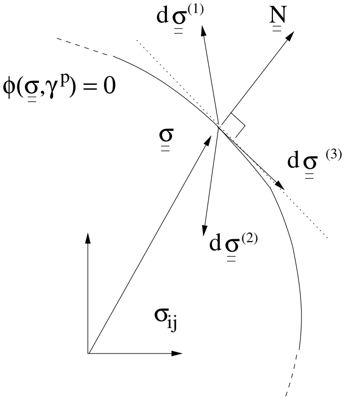
There is thus orthogonality between $d\underline{\underline{\sigma}}$ and $\nabla_\sigma \phi$

Conclusion : $\nabla_\sigma \phi$ is normal to the yield surface.

Denoted $\underline{\underline{N}}$ or $\partial \phi / \partial \underline{\underline{\sigma}}$.

Part II : Multi-axial plasticity theory

The normal to the elastic domain



How to define the loading conditions ?

- Plastic flow requires : $\phi(\underline{\sigma}, \gamma^p) = 0$

but the stress increment must be oriented towards the exterior of the elastic domain.
Example 1 : $\underline{\underline{\sigma}}^{(1)} : \underline{\underline{N}} > 0$ YES, plastic flow,
Example 2 : $\underline{\underline{\sigma}}^{(2)} : \underline{\underline{N}} < 0$ NO, elastic unloading.
Example 3 : $\underline{\underline{\sigma}}^{(3)} : \underline{\underline{N}} = 0$ NO, neutral loading.

Conclusion : The stess point is on the criterion

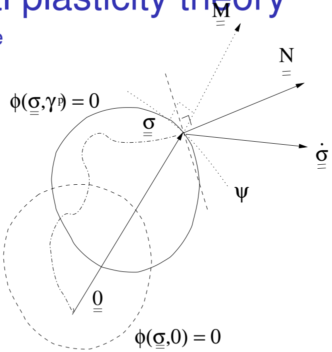
$$\phi(\underline{\underline{\sigma}}, \gamma^p) = 0\,, \quad \text{and} \quad \underline{\underline{N}} : \dot{\underline{\underline{\sigma}}} > 0\,. \tag{12}$$

Elastic unloading and neutral loading

$$\begin{aligned} \phi \leq 0\,, \quad & \underline{\underline{N}} : \dot{\underline{\underline{\sigma}}} = 0\,, \\ \phi = 0\,, \quad & \underline{\underline{N}} : \dot{\underline{\underline{\sigma}}} < 0\,. \end{aligned} \tag{13}$$

Part II : Multi-axial plasticity theory

Work-hardening in stress space



- The example is for kinematics hardening.
- A second potential Ψ of normal $\underline{\underline{M}}$ for the flow rule :

$$\underline{\underline{\dot{\epsilon}}}^p = \dot{\gamma}^p \underline{\underline{M}}, \quad \underline{\underline{M}} = \frac{\partial \psi}{\partial \underline{\underline{\sigma}}}, \tag{14}$$

Same potential $\Psi = \phi$, associated plasticity (eg. metal plasticity)
Different potentials $\Psi \neq \phi$, non-associated plasticity (soils, rocks,...)

•Consistency condition during continuous plastic flow :

$$\dot{\phi}(\underline{\underline{\sigma}}, \gamma^p) \equiv \nabla_\sigma \phi : \underline{\underline{\dot{\sigma}}} + \frac{\partial \phi}{\partial \gamma^p} \dot{\gamma}^p = \underline{\underline{N}} : \underline{\underline{\dot{\sigma}}} - h \dot{\gamma}^p = 0 \ . \tag{15}$$

• Decomposition of strain rate in elastic and plastic parts

$$\underline{\underline{\dot{\epsilon}}} = \underline{\underline{\dot{\epsilon}}}^e + \underline{\underline{\dot{\epsilon}}}^p \, . \tag{16}$$

• Hooke's law :

$$\underline{\underline{\dot{\sigma}}} = \mathbb{S}^e : (\underline{\underline{\dot{\epsilon}}} - \underline{\underline{\dot{\epsilon}}}^p) \, . \tag{17}$$

Isotropic case :

$$\mathbb{S}^e = \lambda \underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + 2\mu \mathbb{I}_S \, , \tag{18}$$

where λ and μ are Lamé constants (Pa).

Part II : Multi-axial plasticity theory

Further generalisation

- Define $\dot{\gamma}^p$ in terms deformation rate :

$$\underline{\underline{N}} : \mathbb{S}^e : (\underline{\underline{\dot{\epsilon}}} - \underline{\underline{\dot{\epsilon}}}^p) - h\dot{\gamma}^p = 0 \; .$$

$$\underline{\underline{N}} : \mathbb{S}^e : \underline{\underline{\dot{\epsilon}}} = \dot{\gamma}^p \left(\underline{\underline{N}} : \mathbb{S}^e : \underline{\underline{M}} + h \right) \; .$$

$$\dot{\gamma}^p = \frac{1}{H} \underline{\underline{N}} : \mathbb{S} : \underline{\underline{\dot{\epsilon}}} \, , \quad H = \underline{\underline{N}} : \mathbb{S} : \underline{\underline{M}} + h \, , \quad \left(H = N_{ij} \mathbb{S}_{ijkl} M_{lk} + h \right) . \quad (19)$$

- Construction of the elasto-plastic tangent operator :

$$\underline{\underline{\dot{\sigma}}} = \mathbb{S}^e : (\underline{\underline{\dot{\epsilon}}} - \dot{\gamma}^p \underline{\underline{M}}) \, .$$

$$\underline{\underline{\dot{\sigma}}} = \mathbb{S}^{ep} : \underline{\underline{\dot{\epsilon}}} \, , \quad \mathbb{S}^{ep} = \mathbb{S}^e - \frac{1}{H} \mathbb{S}^e : \underline{\underline{M}} \otimes \underline{\underline{N}} : \mathbb{S}^e \, , \quad (20)$$

$$\left(\mathbb{S}^{ep}_{ijkl} = \mathbb{S}^e_{ijkl} - \frac{1}{H} \mathbb{S}^e_{ijpq} M_{qp} N_{mn} \mathbb{S}^e_{nmkl} \right) .$$

Part III :

Examples of elastic domain

- Von Mises, Tresca
- Mohr-Coulomb, Drucker-Prager
- Why non-associated plasticity ?

Part III : Examples of elastic domain

Stress invariants

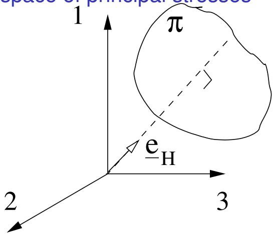
- Stress invariants :

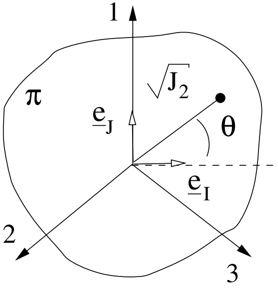
$$
\begin{aligned}
I_1 &= \mathrm{tr}(\underline{\underline{\sigma}})\,, \\
I_2 &= \frac{1}{2}\Big(\mathrm{tr}(\underline{\underline{\sigma}})^2 - \mathrm{tr}(\underline{\underline{\sigma}}^2)\Big)\,, \\
I_3 &= \det(\underline{\underline{\sigma}})\,.
\end{aligned}
$$

- Deviatory stress : $\underline{\underline{\sigma}}^{\mathrm{dev}} = \underline{\underline{\sigma}} - \frac{1}{3} I_1 \underline{\underline{\delta}}$
- $J_2 \equiv \frac{3}{2}\mathrm{tr}(\underline{\underline{\sigma}}^{\mathrm{dev}\,2}) = I_1^2 - 3 I_2$
- Isotropic material : $\phi(I_1, I_2, I_3)$ by symmetry.

Part III : Examples of elastic domain

π plane in space of principal stresses

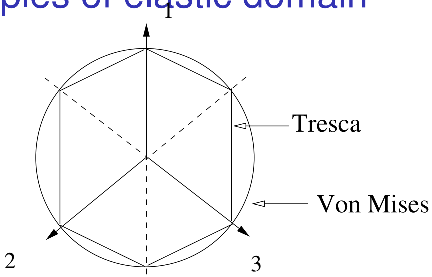




- The hydrostatic direction $\underline{e}_H$
- The π plane is perpendicular to the hydrostatic direction.
- The stress point is decomposed as $\sigma_H \underline{e}_H + \sigma_I \underline{e}_I + \sigma_J \underline{e}_J$
- Introduce Lode's angle θ such that : $\tan\theta = \sigma_J / \sigma_I$.
- The stress point is represented by 3 scalars : $\sigma_H, \sqrt{J_2}, \theta$.

Part III : Examples of elastic domain

Von Mises and Tresca



- Tresca : $\text{Max}(\sigma_i - \sigma_j) \leq \sigma_T$

- Von Mises (1913) : $\sqrt{J_2} \leq \sigma_{VM}$

σ_{VM} is the material yield in a tensile test.

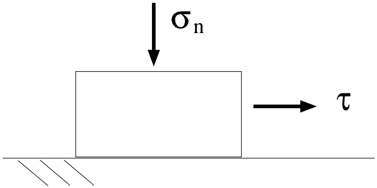
- No dependence on σ_H.

- The two criteria have common pts in the π plane if $\sigma_{VM} = \sqrt{3}\sigma_T$

σ_{VM} and σ_T are material properties

Part III : Examples of elastic domain

Coulomb material



- For an interface : $\tau + \tan\phi\sigma_n \leq C$

with ϕ the friction angle and C, the cohesion ;

also $\sigma_n = \underline{n} \cdot (\underline{\underline{\sigma}} \cdot \underline{n})$, $\tau = |\underline{\tau}|$ and $\underline{\tau} = (\underline{\underline{\delta}} - \underline{n} \otimes \underline{n}) \cdot \underline{T}$

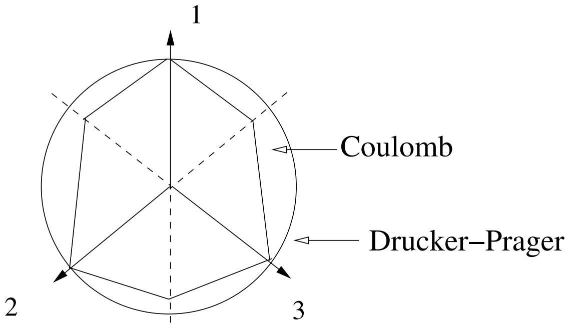
- If the fault orientation is arbitrary (pristine material)

$$\sigma_1(1 - \sin\phi) - \sigma_3(1 + \sin\phi) \leq 2C\cos\phi$$

- No dependence on intermediate principal stress.

Part III : Examples of elastic domain

Coulomb material



- Drucker-Prager : $\alpha_{DP} I_1 + \sqrt{\frac{1}{3} J_2} - C_{DP} \leq 0$

with $\alpha_{DP} = \frac{\tan\phi}{\sqrt{9+12\tan^2\phi}}$ and $C_{DP} = \frac{3C}{\sqrt{9+12\tan^2\phi}}$

Part III : Examples of elastic domain

What is the flow potential ?

- Recall flow rule : $\underline{\underline{\dot{\epsilon}}}^p = \dot{\gamma}^p \underline{\underline{M}}$ with $\underline{\underline{M}} = \frac{\partial \psi}{\partial \underline{\underline{\sigma}}}$.
- Why is $\underline{\underline{M}}$ different from $\underline{\underline{N}} = \frac{\partial \phi}{\partial \underline{\underline{\sigma}}}$ where ϕ is the yield surface.
- Consider Von Mises yield criterion :

$$dJ_2 = \frac{1}{2J_2}\frac{3}{2}2\underline{\underline{\sigma}}^{\mathrm{dev}} : d\underline{\underline{\sigma}}^{\mathrm{dev}}$$

$$\underline{\underline{\sigma}}^{\mathrm{dev}} : d\underline{\underline{\sigma}}^{\mathrm{dev}} = \underline{\underline{\sigma}}^{\mathrm{dev}} : (d\underline{\underline{\sigma}} - \frac{1}{3}\underline{\underline{\delta}} dI_1 = \underline{\underline{\sigma}}^{\mathrm{dev}} : d\underline{\underline{\sigma}}$$

$$dJ_2 = \frac{3}{2J_2}\underline{\underline{\sigma}}^{\mathrm{dev}} : d\underline{\underline{\sigma}}$$

so

$$\underline{\underline{N}} = \frac{3}{2J_2}\underline{\underline{\sigma}}^{\mathrm{dev}}$$

If $\psi = \phi$ then $\underline{\underline{N}} = \underline{\underline{M}}$ and $\mathrm{tr}(\underline{\underline{\dot{\epsilon}}}^p) = 0$
OK for metals but not for rock and soils.

Part III : Examples of elastic domain

What is the flow potential ?

- Consider Drucker-Prager : $\underline{\underline{N}} = \alpha_{DP}\underline{\underline{\delta}} + \frac{\sqrt{3}}{2J_2}\underline{\underline{\sigma}}^{\mathrm{dev}}$

- If associated plasticity : $\mathrm{tr}(\underline{\underline{\dot{\epsilon}}}^p) = \dot{\gamma}^p 3\alpha_{DP}$

and thus change of volume defined by the friction angle ?

- *Conclusion* : need another potential to compute the material dilatancy.

- Is this always true ? Not really, looks at the Cam-Clay model and the critical state theory !

The plot thickens ...

Part III :

Strain localization

- The onset of the first discontinuity
- The pseudo-acoustic tensor
- The analogy with the real acoustic tensor.

Part IV : Strain localization

The onset of the first discontinuity

- First discontinuity : $[\![\underline{u}]\!] = \underline{0}$, $[\![\underline{\dot{u}}]\!] = \underline{0}$,

$$[\![\underline{\underline{\nabla \dot{u}}}]\!] = \underline{\dot{q}} \otimes \underline{n}$$

with $\underline{n}$ normal to surface and $\underline{\dot{q}}$ polarization-like vector.

- Mechanical equilibrium : $[\![\underline{\dot{T}}]\!] = \underline{0}$,

where $\underline{T} = \underline{\underline{\sigma}} \cdot \underline{n}$, the stress vector.

- Material response : $\underline{\underline{\dot{\sigma}}} = \mathbb{S}^{ep} : \underline{\underline{\dot{\epsilon}}}$

assume plastic loading on the two sides on the discontinuity

(comparison solid of Hill)

Part IV : Strain localization

The onset of the first discontinuity

- Combine :

$$[\![\underline{\underline{\dot{\sigma}}}]\!] \cdot \underline{n} = \underline{0}$$

$$\left(\mathbb{S}^{ep} : [\![\underline{\underline{\dot{\epsilon}}}]\!]\right) \cdot \underline{n} = \underline{0}$$

$$\left(\mathbb{S}^{ep} : \underline{\dot{q}} \otimes \underline{n}\right) \cdot \underline{n} = \underline{0}$$

- Introduce the pseudo-acoustic tensor $\underline{\underline{A}}^{ep}(\underline{n})$ with $A^{ep}_{ik} = \mathbb{S}^{ep}_{ijkl} n_j n_k$
- First discontinuity when :

$$\underline{\underline{A}}^{ep}(\underline{n}) \cdot \underline{\dot{q}} = \underline{0}$$

Search for the first zero-eigenvalue, the corresponding eigenvector is the polarization-like vector.

Part V : References

• Davis R.O and Selvadurai A.P.S, Plasticity and geomechanics, Cmabridge University Press, 2002.