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Enhancing the Bolzon–Schrefler–Zienkiewicz Constitutive Model for Partially Saturated Soil

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Abstract. This paper presents an enhanced version of the elasto-plastic model for partially saturated soil first proposed by Bolzon, Schrefler and Zienkiewicz in 1996, "BSZ" model, which uses the effective stress tensor and suction as independent stress variables. It is recalled that the effective stress tensor proposed by Lewis and Schrefler in 1982 is thermodynamically consistent and, compared with other choices of stress tensors, results particularly suitable for partially saturated soil mechanics. A hydraulic constitutive relationship and a hydraulic hysteresis are introduced in the model, to take into account the irreversible deformation during cyclic drying and wetting until structural collapse. For this reason the plastic rate of strain is split into the sum of two components: one depending on the effective stress tensor and the other one on suction. This is the new feature of the BSZ model. This enhanced model is then cast into a thermodynamical framework at macroscopic level and it is shown that it is possible to derive the constitutive law from the Helmholtz free energy and a dissipation function, both for associative and non-associative plasticity. Finally the model predictions have been compared with experimental data for Sion slime, with particular emphasis on the deviatoric part, and model predictions of hysteretic behaviour have been investigated in case of a wetting and drying cycle on compacted betonite–kaolin.

Key words: constitutive relation, partial saturation, plasticity, hydraulic hysteresis, thermodynamics, Helmholtz free energy, dissipation function.

1. Introduction

Several constitutive models have been developed recently to describe the behaviour of partially saturated soils. The choice of the stress tensor appeared immediately to be of primary importance (Fredlund and Morgenstern, 1977). In most elasto-plastic models the yield surface is a function of two stress variables. One choice is the net stress in combination with suction, see Alonso *et al.* (1990), Cui *et al.* (1995), Wheeler and Sivakumar (1995), Vaunat *et al.* (2000) and Buisson and Wheeler (2000). A drawback of this

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choice is that the stress tensor changes form between saturated and unsaturated states; when the pore air is absent, the constitutive equations for saturated states cannot recover those for unsaturated states without additional control (Sheng *et al.*, 2004).

Another choice is the effective stress as defined below by Equation (1), also improperly called Bishop stress, in conjunction with suction. These two variables are used by Jommi and Di Prisco (1994), Bolzon *et al.* (1996), Zienkiewicz *et al.* (1999), Hofstetter *et al.* (1999), Gallipoli *et al.* (2003), Sheng *et al.* (2003a, b), Eberhardsteiner *et al.* (2003), Borja (2004), Tamagnini and Pastor (2004), Sheng *et al.* (2004), and Ehlers *et al.* (2004). With this choice the change from fully saturated states to partially saturated ones is straight forward. Then there is the effective stress tensor defined by Coussy (1995), which, together with suction, is used by Fleureau *et al.* (2003), Hoxa *et al.* (2004), and Mounajed and Obeid (2004). Actually there is a second version of this stress tensor, Coussy (2004), the differential of which however coincides with the first stress tensor. And the first stress tensor has been obtained by lumping the face interfaces together with the solid grains, i.e. the interfaces do not have independent thermodynamic properties.

The respective merits of the stress tensors by Lewis and Schrefler (1982) and by Coussy (1995) are discussed in Gawin and Schrefler (1996) and in Chateau and Dormieux (2002).

Other models (Kogho *et al.*, 1993: Modaressi and Abou-Bekr, 1994: Khalili *et al.*, 2000) are developed with only one stress variable which usually is a combination of some of the above-mentioned stress measures.

The purpose of this paper is twofold: first the model originally developed by Bolzon *et al.* (1996) for partially saturated geomaterials as extension of a generalized plasticity model for fully saturated soils is enhanced. This model, in the sequel referred to as BSZ (Bolzon– Schrefler–Zienkiewicz) model, makes use of the effective stress tensor of Equation (1) and of suction as stress variables. It considers volumetric and deviatoric strain hardening and takes account of the memory of past stress history by suction effects on the stiffness.

A new feature introduced here is the dependence of the plastic strain rate on suction. Jommi and Di Prisco (1994) and Tamagnini and Pastor (2004) have proposed such a dependence by defining the total strain rate as a sum of three components: the elastic strain tensor, the plastic strain tensor coupled with the effective stress tensor, and the plastic strain tensor coupled with suction. This dependence of the plastic strain rate on suction takes into account irreversible deformation during cyclic drying and wetting, when the suction increases or decreases, until the structural collapse, while the effective stress remains constant.

The second aspect is the theoretical validation of the enhanced BSZ model through the laws of thermodynamics: the model must obey thermodynamic principles and its constitutive equations are derived from them. For this purpose thermodynamic relations postulated at macroscale are used.

A formulation of elasto-plastic theory for rate independent materials in general is based on the use of thermodynamic potentials, i.e. the dissipation potential and one of the four energy functions, internal energy, Helmholtz free energy, enthalpy and Gibbs free energy. The energies are function of the independent variables and the Legendre transformations are used to establish the links between them.

The procedure employed follows the work of Ziegler (1983), Collins and Houlsby (1997) and Houlsby and Puzrin (2000). This procedure shows how the plasticity theories can be developed after stating the thermodynamics assumptions. From an energy function and a dissipation function that obey thermodynamic laws, it is possible to derive the constitutive laws of a model. The Helmholtz free energy and the dissipation potential are mainly used. These potential functions are defined for the case of associative and non-associative plasticity. This last one refers to "frictional materials" (Collins and Houlsby, 1997).

Finally, an experimental validation of this model is presented. Theoretical predictions of the model in drained triaxial tests are made and compared with experimental observations of Sion slime (Geiser, 1999), for the saturated and unsaturated case. The hysteretic behaviour predictions of the model are investigated in case of wetting and drying cycles on compacted betonite–kaolin (Sharma, 1998).

2. Enhanced BSZ Constitutive Model

2.1. STRESS VARIABLES

As already stated, it is known from experiments that two stress variables are necessary to completely describe the behaviour of partially saturated soil, including collapsible behaviour. These stress variables should be thermodynamically and mechanically consistent. Mechanical consistency means that the selected stress and strain variables should be work conjugate. Houlsby (1997) has shown that the following pair:

effective stress in the form

$$\sigma'_{ij} = \sigma_{ij} + \delta_{ij}(S_w p^w + S_g p^g) \tag{1}$$

and capillary pressure, also called suction

$$s = p^c = p^g - p^w \tag{2}$$

scaled by porosity, n (modified suction, ns) are work-conjugate respectively with the solid strain ε_{ij} and the degree of saturation S_w. In the above equations σ_{ij} is the total stress tensor, σ'_{ij} is the "effective" stress responsible for deformations, δ_{ij} the Kronecker symbol and p^w and p^g are the water and gas pressures, respectively. Traction stresses and compressive pressures are here assumed as positive.

The form (1) was first introduced by Lewis and Schrefler (1982) while extending Biot's theory to two-phase flow in a deforming porous medium. It was derived again by Hassanizadeh and Gray (1990) and Gray and Hassanizadeh (1991) using an averaging approach and taking into account the interfaces endowed with thermodynamic properties. The consistency of both above indicated variables with the thermodynamic forms assumed was shown by these authors.

The above stress variables were chosen in the BSZ model (1996). The thermodynamic consistency of the model itself had not been shown yet. This is carried out here by considering thermodynamics at macroscale, but on an enhanced version of the BSZ model which incorporates some ideas put forward by Gens (1995), Sheng *et al.* (2004), Tamagnini and Pastor (2004). The proof of thermodynamic consistency of the original BSZ model can be found in Santagiuliana (2004). The enhanced model is shown next.

2.2. ELASTO-PLASTIC MODEL

In standard elasto-plasticity, the total strain rate $\dot{\varepsilon}$ is divided into an elastic component $\dot{\varepsilon}^e$ and a plastic component $\dot{\varepsilon}^p$.

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p. \tag{3}$$

The elastic strain rate is related to the stress rate by an injective relation

$$\dot{\sigma}_{ij} = D^e_{t\ ijkl}\,\dot{\varepsilon}^e_{kl} \tag{4}$$

with D^e_t the tangent elastic stiffness matrix.

Plastic strains can occur when the yield condition is satisfied:

$$f = 0. \tag{5}$$

The direction n_{ij}, i.e. the gradient of the yield function with respect to the stresses, is

$$n_{ij} = \frac{\partial f}{\partial \sigma_{ij}}. \tag{6}$$

The evolution of the plastic strain rates can be modelled by a flow rule

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda} m_{ij} \tag{7}$$

with $\dot{\lambda}$ the consistency parameter and with m_{ij} the normal to a plastic potential function g

$$m_{ij} = \frac{\partial g}{\partial \sigma_{ij}}. \tag{8}$$

In the case that the plastic potential function is equal to the yield function f, plastic flow is called associated and the plastic strain rates are then orthogonal to the yield surface f.

In generalized plasticity the Equations (3) and (4) still hold, together with the flow rule (7) where m_{ij} can be freely chosen for loading or unloading, i.e. it does not necessarily follow from a plastic potential function. However, as remarked by de Borst and Heeres (2000), generalized plasticity does not require, but also does not preclude the existence of yield functions, plastic potential functions or stability postulates. A generalized plasticity model is determined by specifying the three directions n_{ij}, $m_{\mathrm{L}ij}$ and $m_{\mathrm{U}ij}$ as well as the hardening moduli H_{L} and H_{U}. The subscripts L/U are used to distinguish between loading and unloading.

For this purpose one can employ a surface or the above quantities can be defined without reference to any surface.

We proceed as follows: m_{ij} and n_{ij} are defined and the surfaces f and g are then obtained by integration. Since we have these surfaces, we replace the loading–unloading conditions of generalized plasticity by the Kuhn–Tucker relations to determine if the stress point is on the yield surface. The consistency parameter $\dot{\lambda}$ is here further simply defined (Pastor *et al.*, 1990), in a form similar to standard plasticity, because consistency is not enforced in generalized plasticity:

$$\dot{\lambda} = \frac{n_{ij}{}^{\mathrm{T}} D^{e}_{t\ ijmn}\ \dot{\varepsilon}_{mn}}{h_{\mathrm{L/U}} + n_{ij}{}^{\mathrm{T}} D^{e}_{t\ ijkl}\ m_{\mathrm{L/U}kl}} \tag{9}$$

for loading and unloading, respectively; h is the hardening modulus.

The BSZ model is presented here following the paper of Bolzon *et al.* (1996), but more importance is given to the derivation of the yield and potential surfaces, because later on it is shown how these can be derived from the thermodynamics laws.

As in Pastor *et al.* (1990) and in Bolzon *et al.* (1996), stress states can be investigated in the space of effective mean stress, p' and deviatoric stress q, defined as

$$p' = -\frac{tr\sigma_{ij}'}{3}, \tag{10}$$

$$q = \sqrt{\frac{3}{2} dev(\sigma_{ij}') : dev(\sigma_{ij}')}, \tag{11}$$

where the signs have been chosen to obtain positive values of p' and $dev(\sigma_{ij}{}') = \sigma'_{ij} - p'\delta_{ij}$.

The model derived within the framework of generalized plasticity makes use of the loading–unloading direction vector and the direction vector defining the plastic flow:

the loading–unloading direction vector

$$n_{ij} \equiv \frac{1}{\sqrt{1+d_f^2}} \left\{d_f; 1\right\}^T, \tag{12a}$$

$$d_f = (1+c)(M_f - \eta), \tag{12b}$$

the direction vector defining the plastic flow

$$m_{ij} \equiv \frac{1}{\sqrt{1+d_g^2}} \left\{d_g; 1\right\}^T, \tag{13a}$$

$$d_g = (1+c)(M_g - \eta), \tag{13b}$$

where c is a material parameter independent of suction, M_f, M_g are material parameters i.e. the slope of the critical state line and the slope of the line defining zero dilatancy in (p', q) space, and η is the stress ratio: $\eta = \dfrac{q}{p'}$. For more details, see Pastor *et al.* (1990).

By integration of Equations (12.a) and (13.a), as suggested in Pastor *et al.* (1990) we obtain the yield surface f and the potential surface g:

$$f \equiv q - M_f p' \left(1 + \frac{1}{c}\right)\left[1 - \left(\frac{p'}{p_f}\right)^c\right], \tag{14}$$

$$g \equiv q - M_g p' \left(1 + \frac{1}{c}\right)\left[1 - \left(\frac{p'}{p_g}\right)^c\right], \tag{15}$$

where the parameters p_f and p_g depend on suction.

For associative plasticity $n_{ij} = m_{ij}$, so the yield and potential surfaces coincide, $f = g$ and the formulation is simplified.

2.3. HYDRAULIC BEHAVIOUR

An important feature of unsaturated soils is the hydraulic hysteresis, that can be recorded during cycles of wetting and drying.

From experimental and theoretical evidence, the relationship between suction and degree of water saturation depends on the history of flow. This dependence is known as hysteresis.

Hysteresis is here accounted for to describe inelastic behaviour during cyclic wetting–drying tests and structural collapse, when the bonding forces due to capillarity are removed by wetting.

We introduce in the model two additional yield surfaces defining the elastic and elasto-plastic drying and wetting boundaries like in Sheng *et al.* (2004). These surfaces are denoted by the index SI for suction-increase (drying) yield surface and SD for the suction-decrease (wetting) yield surface. In the (p', s) plane (Figure 1) these are two parallel horizontal lines that move upwards (or downwards) for the stress paths, where the suction increases (or decreases) while the effective mean stress remains constant. The new two surfaces SI and SD describe the irreversible changes in volume and degree of water saturation caused by cyclic drying and wetting under constant effective stress.

In Figures 1 and 2 the intersection of the yield surface with the plane (p', s), and the (p', q) plane is shown while Figure 3 depicts the surface in the (p', q, s) space.

In the (s, S_w) plane any cycle of drying and wetting has to be within two limiting curves obtained by drying from a fully saturated state, ψ_{SI}, and wetting from a dry state, ψ_{SD}. This condition is shown in Figure 4. The functions ψ_{SI} and ψ_{SD} are suggested by Romero and Vaunat (2000), based on the water retention equation of van Genuchten (1980). These two functions link the suction to the degree of water saturation with an elasto-plastic relation that depends on water pressure. Neglecting the hysteresis phenomenon, the same curve can be used for wetting and drying paths.

The elastic relation between ns and S_w is assumed to be linear:

$$\frac{\mathrm{d}S_w^e}{n\mathrm{d}s} = K_s. \tag{16}$$

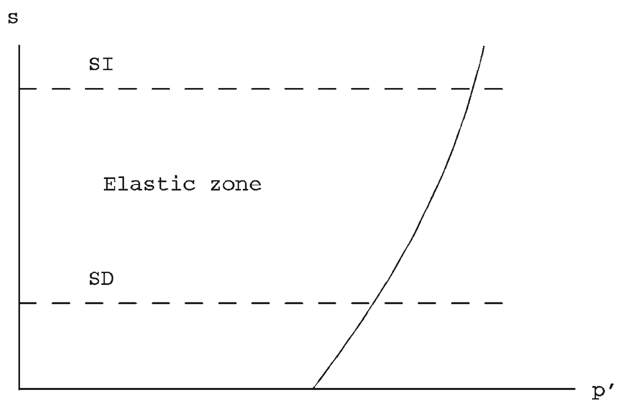

Figure 1. Yield surface in (p', s) plane.

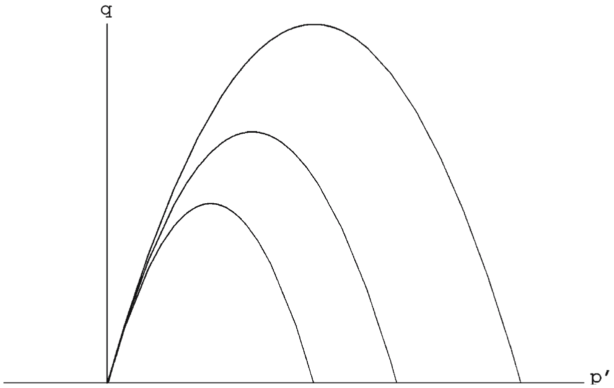

Figure 2. Yield surface in (p', q) plane.

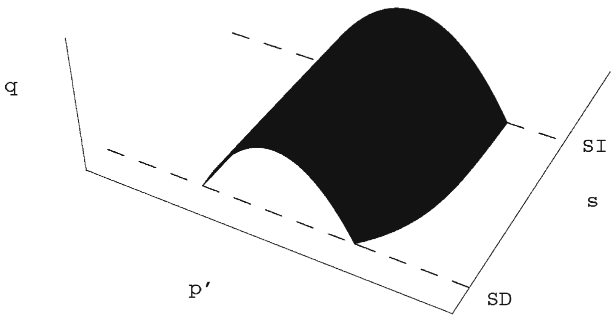

Figure 3. Yield surface in (p', q, s) space.

The relation between the plastic increment of S_w and the modified suction is defined as

$$\mathrm{d}S_w^p = \left(\frac{\mathrm{d}\psi_\alpha}{n\mathrm{d}s} - \frac{1}{K_s}\right)\mathrm{d}s \tag{17}$$

with $\alpha = \mathrm{SI}, \mathrm{SD}$ and K_s derives from the scanning curve (16).

There is one condition between ψ_α and K_s:

$$\left.\frac{\mathrm{d}\psi_{\mathrm{SI}}}{\mathrm{d}s}\right|_{s=0} \leqslant \frac{1}{K_s} \leqslant \left.\frac{\mathrm{d}\psi_{\mathrm{SD}}}{\mathrm{d}s}\right|_{s=0} \tag{18}$$

This condition ensures that the scanning curves are inside the main drying and wetting curves.

As in Sheng *et al.* (2004), using ns instead of s provides some advantage in the thermomechanical considerations of the model, but it is generally not necessary in deriving the constitutive equations. There is no essential difference in these two stress quantities, as the porosity n plays a role similar to a scaling parameter.

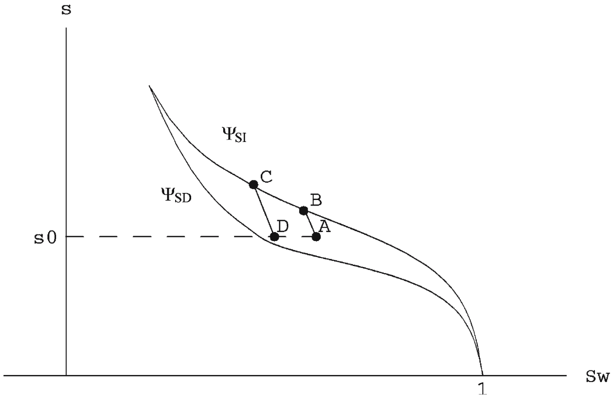

Figure 4. Hydraulic behaviour under constant void ratio.

The scanning curve is an elastic path from a general point A (Figure 4) until the suction reaches the main drying or wetting curve ψ_{SI}, ψ_{SD}. For example, drying from a general state, point A (S_{wA}, $s0$), the suction reaches the main drying curve at point B by a scanning curve. Continued drying follows now the main drying curve ψ_{SI}, e.g. until point C. If the soil is wetted to initial suction, it follows the elastic scanning curve to point D. The difference in the degree of water saturation between point A and D is due to the hydraulic hysteresis.

The evolution of SI and SD yield surfaces is controlled by the yielding suction s_{I} and s_{D} for drying and wetting respectively.

$$\mathrm{d}s_\alpha = \frac{\mathrm{d}s_\alpha}{\mathrm{d}S_w^p}\mathrm{d}S_w^p = \left(\frac{K_s}{\dfrac{\mathrm{d}\psi_\alpha}{\mathrm{d}s}K_s - 1}\right)\left(\mathrm{d}\lambda_\alpha \frac{\partial g_\alpha}{\partial s}\right), \quad \alpha = I, D, \tag{19}$$

where $\mathrm{d}\lambda_\alpha$ are plastic multipliers defined as in Sheng *et al.* (2004)

$$\mathrm{d}\lambda_\alpha = \frac{K_s \dfrac{\partial f_\alpha}{\partial s}\mathrm{d}S_r}{-\dfrac{\partial f_\alpha}{\partial s_\alpha}\dfrac{\mathrm{d}s_\alpha}{\mathrm{d}S_w}\dfrac{\mathrm{d}g_\alpha}{\mathrm{d}s} + K_s \dfrac{\partial f_\alpha}{\partial s}\dfrac{\mathrm{d}g_\alpha}{\mathrm{d}s}} \tag{20}$$

with f_α yield surfaces:

$$f_{\mathrm{SI}} = s - s_I, \tag{21}$$

$$f_{\mathrm{SD}} = s_D - s \tag{22}$$

and g_α the plastic potential surfaces that are different from the SI and SD yield surfaces, in case of non-associated flow rules.

2.4. INCREMENTAL STRESS-STRAIN EQUATIONS

As in Bolzon *et al.* (1996), the volumetric strain, work-conjugate with p', is defined

$$\varepsilon_v = -\mathrm{tr}\varepsilon_{ij}, \tag{23}$$

and the deviatoric strain work-conjugate with q:

$$\varepsilon_s = \frac{2}{3}\sqrt{\frac{1}{2}\mathrm{dev}(\varepsilon_{ij}):\mathrm{dev}(\varepsilon_{ij})}. \tag{24}$$

The volumetric strain ε_v is positive in compression. The strain increment is decomposed into an elastic part and an elasto-plastic one, see Equation (3).

It is assumed that during elastic unloading, with constant suction, the volumetric strain increment corresponds to the elastic strain increment which depends on stress increment:

$$\mathrm{d}\varepsilon_v = \mathrm{d}\varepsilon_v^e = \frac{k\mathrm{d}p'}{v_0 p'}, \tag{25}$$

k is a constant independent of suction and v_0 is the initial specific volume.

Further during plastic loading, the volumetric strain increment is composed of an elastic and a plastic part and can be expressed with constant suction:

$$\mathrm{d}\varepsilon_v = \mathrm{d}\varepsilon_v^e + \mathrm{d}\varepsilon_v^p = \frac{\bar{\lambda}\mathrm{d}p'}{v_0 p'}, \tag{26}$$

where $\bar{\lambda}$, the soil compressibility, is a parameter dependent on suction.

The elastic deviatoric strain increment is defined as

$$\mathrm{d}\varepsilon_s^e = \frac{1}{G}\mathrm{d}q, \tag{27}$$

where G is the shear modulus.

The effect of suction is here taken into account in the constitutive equation for partially saturated soils through the introduction of a suction dependent term as in the formulation of constitutive models for unsaturated soils by Tamagnini and Pastor (2004) and Jommi and Di Prisco (1994).

The plastic part of the rate of strains in unsaturated soil is obtained from the sum of two terms:

$$\dot{\varepsilon}_{ij}^p = \frac{1}{H}\dot{\sigma}'_{kl} n_{gij} n_{fkl} + \frac{1}{H_b}\dot{s} n_{gij}. \tag{28}$$

The first term is similar to the hardening law valid for saturated soil, but it depends on suction.

The second term refers only to unsaturated soil and imposes the dependence of plastic strain also on changes in suction during wetting.

H is the plastic modulus which depends on the material characteristics and explicit expressions for clays and sands are given by Pastor *et al.* (1990) and Zienkiewicz *et al.* (1985).

For generic stress paths and for deviatoric effects the plastic modulus results in

$$H = H_0 p' H_w H_f (H_v + H_s) H_{\mathrm{d}m}, \tag{29}$$

where

$$H_0 = \frac{1 + e_0}{\bar{\lambda}(0) - k}, \tag{30}$$

e_0 is the initial void ratio, related to the initial specific volume by $v_0 = 1 + e_0$.

$$H_w = 1 + as, \quad a \text{ is a material parameter}, \tag{31}$$

$$H_f (H_v + H_s) \text{ depends on deviatoric strain}, \tag{32}$$

$$H_s = \beta_0 \beta_1 \exp(-\beta_0 \xi) \tag{33}$$

with the accumulated plastic strain

$$\xi = \int \mathrm{d}\xi = \int \left| \frac{\mathrm{d}\varepsilon_s^p}{\mathrm{d}\tau} \right| \mathrm{d}\tau \tag{34}$$

in which $\dfrac{\mathrm{d}\varepsilon_s^p}{\mathrm{d}\tau} = \dot{\varepsilon}_s^p$,

$$H_f = \left(1 - \frac{\eta}{\eta_f}\right) \quad \text{with} \tag{35}$$

$$\eta_f = \left(1 + \frac{1}{c}\right) M_f, \tag{36}$$

$$H_v = \left(1 - \frac{\eta}{M_g}\right), \tag{37}$$

$$H_{dm} = \left(\frac{\zeta_{\max} J(s)}{\zeta}\right)^{\gamma}. \tag{38}$$

ζ is the mobilized stress function defined as

$$\zeta = p'\left[1 - \left(\frac{1}{1+c}\right)\frac{\eta}{M}\right]^{\frac{1}{c}} \tag{39}$$

and $\zeta_{\max}$ is the maximum value reached by function ζ.

J provides an additional form of hardening due to partial saturation and is stated as

$$J(s) = \mathrm{e}^{[\alpha(1-S_w)]}, \tag{40}$$

H_b is defined as

$$H_b = w H_0 p' H_f H_{\mathrm{d}m}, \tag{41}$$

where w can be constant or it can be assumed as to be function of s.

This modulus can be determined starting from a wetting path in which the material undergoes collapse (Tamagnini and Pastor, 2004), because the wetting path is an unloading stress path, hence the first term of Equation (28) is zero, in accordance with the saturated case. This means that isotropic collapse is proportional to the plastic strain in isotropic strain hardening

$$\dot{\varepsilon}_v^p = \frac{1}{w H_0 p' H_f H_{\mathrm{d}m}} \dot{s}. \tag{42}$$

For volumetric deformation during an isotropic virgin compression loading ($\eta = 0$) and $H_s = 0$, $H_{\mathrm{d}m} = H_f = H_v = 1$, at constant suction, the plastic flow takes the form:

$$d\varepsilon_v^p = \frac{1}{H}\frac{dp'}{p'}. \tag{43}$$

This equation is related to Equations (25) and (26) by considering

$$H = \frac{v_0}{\bar{\lambda} - k} \tag{44}$$

as in Bolzon *et al.* (1996).

The dependence on suction of the plastic modulus having been defined, the same effects have to be introduced in the yield and potential surface equations. Experimental observations show that parameter p_f is increasing with suction. Given the initial yield stress p'_{y0i} for saturated conditions, the dependence of p_f on suction is assumed as

$$p_f = p'_{y0i} + is \tag{45}$$

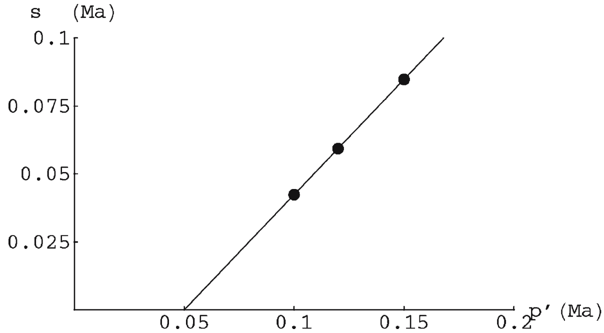

Figure 5. Dependence of p_f on suction in (p', s) plane from experimental data of a partially saturated compacted kaolin (Bolzon *et al.*, 1996).

The parameter i is determined by experimental data, e.g. (Figure 5) for a partially saturated compacted kaolin in Bolzon *et al.* (1996), to obtain an increasing function of suction when water saturation is less than one.

The same variation can be assumed for the parameter p_g: this is suggested by the possibility of assuming associative plasticity, in which case $f = g$.

The evolution of the yield surface, as it is shown in Figure 1, will be governed by

$$p'_y(s) = p'_c \left(\frac{p'_{y0}}{p'_c} \right)^{(\bar{\lambda}(0)-k)/(\bar{\lambda}(s)-k)}, \tag{46}$$

where $p'_y(s)$ is the yield stress at different suction values, p'_{y0} is the initial yield stress for the saturated condition, and p'_c is the reference value of the effective stress, characterizing the limit situation where no plastic strains develop as a consequence of suction changes.

3. Thermodynamic Consistency of the Model

3.1. THERMODYNAMICS FOR UNSATURATED SOILS

As already pointed out, the effective stress σ' and the modified suction ns are work conjugate with the rate of soil skeleton strain $\dot{\varepsilon}$ and the rate of degree of saturation $-\dot{S}^w$, respectively. The suction plays the role of stress, the degree of water saturation is a strain-like variable.

Following Houlsby (1997), Tamagnini and Pastor (2004) and neglecting the mechanical dissipation associated with fluid flow and air compressibility, the rate of work input per unit volume of unsaturated soil is

$$\dot{W} = [\sigma_{ij} + p^g \delta_{ij} - S_w (p^g - p^w) \delta_{ij}] \dot{\varepsilon}_{ij} - n (p^g - p^w) \dot{S}_w = \sigma'_{ij} \dot{\varepsilon}_{ij} - sn \dot{S}_w. \tag{47}$$

To have the thermodynamic formulation of a model we can note (Collins and Houlsby, 1997), that in an isothermal process the rate of work input per unit volume is equal to the sum of the rate of change of free energy $\dot{A}$ and the dissipation D.

$$\dot{W} = \dot{A} + D. \tag{48}$$

The first term in the right-hand side represents the rate of change of recoverable, not necessarily elastic, energy, while the dissipation is the rate at which energy is dissipated. This shows that the two fundamental thermodynamic functions, i.e. the free energy potential and the dissipation function, uniquely define the constitutive model, and all the necessary equations can be derived from them.

The Helmholtz free energy is defined in Schrefler (2002) for the phases and interfaces that compose the porous material. The soil is here investigated in terms of effective stress and solid strains, therefore we use the Helmholtz free energy associated with the solid skeleton for unsaturated soils and this is supposed to depend on the soil skeleton strains ε_{ij}, on kinematic strain-like internal variable α_{ij} and on degree of water saturation

$$A^s(\varepsilon_{ij}, \alpha_{ij}, S_w). \tag{49}$$

The kinematic strain-like internal variable α_{ij} represents the internal changes of the material, e.g. plastic strain or damage, that can sometimes be measured but they cannot be controlled from the outside.

The Helmholtz free energy, for an isothermal process, can be rewritten as

$$A^s = A_1(\varepsilon^e_{ij}, S^e_w) + A_2(\alpha_{ij}, S^p_w) \tag{50}$$

and it follows that, by differentiating the free energy, we obtain

$$\sigma'_{ij} = \frac{\partial A_1}{\partial \varepsilon^e_{ij}} \quad \chi_{ij} = -\frac{\partial A_2}{\partial \alpha_{ij}}, \tag{51}$$

where χ_{ij} is the corresponding stress-like variable, defined also as "dissipation force" or "generalised stress". The dissipation function can be written as

$$D = \chi_{ij}\dot{\alpha}_{ij}. \tag{52}$$

The difference between the true stress tensor σ'_{ij} and the generalized stress tensorχ_{ij} will be denoted by $\rho_{ij} = \sigma'_{ij} - \chi_{ij}$ and called "back stress". (See Collins and Houlsby, 1997).

For a rate independent elastic/plastic deformation, D must be a homogeneous function of degree one in $\dot{\alpha}_{ij}$. Hence, from Euler's theorem for homogeneous functions it follows that

$$D = \frac{\partial D}{\partial \dot{\alpha}_{ij}} \dot{\alpha}_{ij}. \tag{53}$$

From (52) and (53) we have the "orthogonality principle" (see Ziegler, 1983)

$$\chi_{ij} = \frac{\partial D}{\partial \dot{\alpha}_{ij}}. \tag{54}$$

In a reversible process, with $D = 0$, from Equations (47) and (48) follows

$$\dot{W} = \dot{A}^s = \sigma'_{ij} \dot{\varepsilon}_{ij} - sn \dot{S}_w \tag{55}$$

and from this equation results

$$ns = -\frac{\partial A^s}{\partial S_w}, \tag{56}$$

where ns is the measure of changes in the free energy produced by changes in saturation degree.

3.2. THERMODYNAMIC FORMULATION OF THE MODEL

In this model α_{ij} can be identified as the plastic or permanent strain remaining when the stress in the material element returns to its initial value.

The Helmholtz free energy, Equation (50), associated with the solid skeleton now can be written as

$$A^s = A_1(\varepsilon^e_{ij}, S^e_w) + A_2(\varepsilon^p_{ij}, S^p_w) \tag{57}$$

where the second component $A_2(\varepsilon^p_{ij}, S^p_w)$ is the part of the Helmholtz free energy that depends on plastic strains and on the plastic part of saturation degree.

For a general isothermal process the rate of plastic component of work is

$$\dot{W}^p = \dot{A}_2 + D \tag{58}$$

and can be expressed in triaxial stress state terms as suggested by Sheng *et al.* (2004)

$$\dot{W}^p = p' \dot{\varepsilon}^p_v + q \dot{\varepsilon}^p_s - ns \dot{S}^p_w. \tag{59}$$

The enhanced BSZ model, differently from the modified Cam-Clay model, can be generated without the need to introduce a back stress (Collins and Houlsby, 1997) and so all the plastic work due to the first two terms of Equation (59) is dissipated.

The last term in Equation (59) is only relevant to suction increase and suction decrease, the movement of the yield surface f alone does not contribute to S_w^p.

This term is a state function depending only on the current saturation degree value, hence it is integrable and gives zero in a closed loop of the degree of water saturation. According to Equation (58), it can be considered as the second part of the Helmholtz free energy:

$$\dot{A}_2 = -ns\dot{S}_w^p. \tag{60}$$

This equation indicates that suction increase and decrease yield surfaces contribute to the plastic work and the plastic work product is recoverable. During the loading path in unsaturated conditions, the stored energy is recovered by wetting, when the forces due to capillarity are removed; the resulting behaviour is called "structural collapse" (Tamagnini and Pastor, 2004).

In the associative formulation of the BSZ model, $f = g$ and therefore $M_f = M_g$. We consider first this case of associative plasticity.

The dissipation function is generally obtained from Equations (58), (59) and (60) as Equation (61).

$$D = \dot{W}^p - \dot{A}_2 = p'\dot{\varepsilon}_v^p + q\dot{\varepsilon}_s^p. \tag{61}$$

For the BSZ model it is here defined, in associative plasticity, as a function of the volumetric and deviatoric plastic strain rate

$$\begin{aligned} D &= M_f\dot{\varepsilon}_s^p(1+c)p_f\left(\frac{(1+c)\,M_f\dot{\varepsilon}_s^p + c\dot{\varepsilon}_v^p}{(1+c)^2\,M_f\dot{\varepsilon}_s^p}\right)^{(1+\frac{1}{c})} \\ &= M_f\dot{\varepsilon}_s^p(1+c)p'\left(\frac{p'}{p_f}\right)^c. \end{aligned} \tag{62}$$

This is in line with Collins and Houlsby (1997), where from a defined Helmholtz free energy and a defined dissipation function the constitutive laws of the model are derived.

Note that the dissipation, Equation (62), is a homogeneous function of degree one in the plastic strain increments, therefore it coincides with the dissipation potential function.

Hence the stresses p' and q can be derived from the function D:

$$p' = \frac{\partial D}{\partial \dot{\varepsilon}_v^p} = p_f\left(\frac{c}{1+c}\left(\frac{1}{c} + \frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p(1+c)M_f}\right)\right)^{\frac{1}{c}}, \tag{63}$$

$$q = \frac{\partial D}{\partial \dot{\varepsilon}_s^p} = p_f \left(\frac{c}{1+c} \left(\frac{1}{c} + \frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p (1+c) M_f} \right) \right)^{\frac{1}{c}} * \left(M_f - \frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p (1+c)} \right). \tag{64}$$

The associated flow rule for the volumetric and deviatoric part of plastic strain is

$$\dot{\varepsilon}_v^p = \dot{\lambda}(1+c)(M_f - \eta), \quad \dot{\varepsilon}_s^p = \dot{\lambda}, \tag{65}$$

where λ is the plastic multiplier, obtained from Equations (7) and (13). By combining the two Equations (65) and the expression of p' and q (63) (64), we obtain (14), equation that describes the yield function f of the BSZ model.

To have thermodynamical consistency, the dissipation function, that coincides with the dissipation potential function, because of Euler's theorem, has to be always non-negative.

Equation (62) for the dissipation function D is always positive because the material parameters, M, p_f, c, are taken positive with the effective stress tensor p' and the deviatoric part of plastic strain rate increment coincides with the rate of plastic multiplier (65), $\dot{\varepsilon}_s^p = \dot{\lambda}$, so, for the Kuhn–Tucker conditions, $\dot{\lambda} \geqslant 0$, $f(\sigma, k) \leqslant 0$, $\dot{\lambda} f(\sigma, k) = 0$, the deviatoric part of plastic strain rate increment is always non-negative.

Differentiating, instead, the Helmoltz free energy

$$A^s = \frac{k}{v_0} e^{\varepsilon_v^e \frac{v_0}{k}} + \frac{(\varepsilon_s^e)^2}{2} \mathrm{G} - ns S_w, \tag{66}$$

we can also obtain the stress

$$p' = \frac{\partial A^s}{\partial \varepsilon_v^e} \quad q = \frac{\partial A^s}{\partial \varepsilon_s^e} \tag{67}$$

and the suction, as already indicated in Equation (56).

Furthermore, from the Gibbs free-energy expression, obtained from the Helmholtz free energy function, by a Legendre transformation,

$$G^s = \frac{k}{v_0} p'(\ln p' - 1) + \frac{q^2}{2\mathrm{G}} + p' \varepsilon_v^p + q \varepsilon_s^p - ns S_w \tag{68}$$

we can obtain the elastic deformations, because it must be

$$\frac{\partial G^s}{\partial p'} = \varepsilon_v \quad \text{and} \quad \frac{\partial G^s}{\partial q} = \varepsilon_s \tag{69}$$

and particularly

$$\frac{\partial G_1}{\partial p'} = \varepsilon_v^e \quad \text{and} \quad \mathrm{e}\frac{\partial G_1}{\partial q} = \varepsilon_s^e, \tag{70}$$

where

$$G_1(p', q) = \frac{k}{v_0} p'(\ln p' - 1) + \frac{q^2}{2\mathrm{G}}, \tag{71}$$

$$\varepsilon_v^e = \frac{k}{v_0} \ln p' \quad \text{elastic part of volumetric strain,} \tag{72}$$

$$\varepsilon_s^e = \frac{q}{\mathrm{G}} \quad \text{elastic part of shear strain.} \tag{73}$$

The derivation of elastic and plastic relations between the suction and the saturation degree (Equations (16) and (17)) from the energy functions A and G has not yet been developed.

More complex is the case of the non-associative formulation of BSZ model, when $M_f \neq M_g$ and therefore f does not coincide with g.

The non-associated flow rule is

$$\dot{\varepsilon}_v^p = \dot{\lambda}(1+c)(M_g - \eta), \quad \dot{\varepsilon}_s^p = \dot{\lambda} \tag{74}$$

and by arranging these two equations we obtain

$$\frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p} = (1+c)(M_g - \eta). \tag{75}$$

From the yield function, the plastic potential and the non-associated flow rule the expressions of p' and q are

$$p' = p_f \left(\frac{\left(M_f(1+c)^2 - c(1+c)M_g + c\dfrac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p} \right)}{M_f(1+c)^2} \right)^{\frac{1}{c}}, \tag{76}$$

$$q = p_f \left(\frac{\left(M_f(1+c)^2 - c(1+c)M_g + c\dfrac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p} \right)}{M_f(1+c)^2} \right)^{\frac{1}{c}} \times \left(M_g - \frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p(1+c)} \right). \tag{77}$$

For the BSZ model the dissipation function for non associative plasticity is defined as

$$D = p_f \left(\frac{\left(M_f(1+c)^2 - c(1+c)M_g + c\dfrac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p} \right)}{M_f(1+c)^2} \right)^{\frac{1}{c}} \times \\ \times \left[\left(M_g - \frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p(1+c)} \right) * \dot{\varepsilon}_s^p + \dot{\varepsilon}_v^p \right], \tag{78}$$

i.e.

$$D = \dot{\varepsilon}_s^p p'(1+c)\left(M_g - M_f + M_f\left(\frac{p'}{p_f}\right)^c\right). \tag{79}$$

This dissipation function can also be written as

$$D = \dot{\varepsilon}_s^p p'(1+c) M_g\left(\frac{p'}{p_g}\right)^c \tag{80}$$

by using

$$M_f\left(1 - \left(\frac{p'}{p_f}\right)^c\right) = M_g\left(1 - \left(\frac{p'}{p_g}\right)^c\right) \tag{81}$$

derived from the combination of Equation (14) and (15) and by considering that we have plastic flow only when $f = 0$ and $g = 0$.

The expression of the dissipation function (80) is a homogeneous function of degree one in the plastic strain increments, therefore it coincides with the dissipation potential function, because of Euler's theorem.

In the case of non-associative flow, the derivation of the stress tensors p' and q and the generation of constitutive laws from the dissipation function D need the application of the Lagrangian multiplier method (Lippmann, 1972).

As a side condition, according to Equation (75) we will impose a relationship between the volumetric and deviatoric strain rates:

$$\dot{\varepsilon}_v^p - (1+c)\left[M_g - M_f\frac{(1+c)}{c}\left(1 - \left(\frac{p'}{p_f}\right)^c\right)\right]\dot{\varepsilon}_s^p = 0. \tag{82}$$

The standard procedure can now be applied to the modified dissipation function

$$D = \Lambda\dot{\varepsilon}_v^p - \Lambda(1+c)\left[M_g - M_f\frac{(1+c)}{c}\left(1 - \left(\frac{p'}{p_f}\right)^c\right)\right]\dot{\varepsilon}_s^p + \\ + p'(1+c)M_g\left(\frac{p'}{p_g}\right)^c\dot{\varepsilon}_s^p \tag{83}$$

with Λ as the Lagrangian multiplier. Differentiating with respect to the plastic strain rates, gives the components of the generalized stress tensor

$$\chi_p = \frac{\partial D}{\partial \dot{\varepsilon}_v^p} = \Lambda, \tag{84}$$

$$\chi_q = \frac{\partial D}{\partial \dot{\varepsilon}_s^p} = p'(1+c)M_g\left(\frac{p'}{p_g}\right)^c - \\ -\Lambda(1+c)\left[M_g - M_f\frac{(1+c)}{c}\left(1 - \left(\frac{p'}{p_f}\right)^c\right)\right], \tag{85}$$

where χ_p and χ_q are the volumetric and deviatoric part of χ_{ij}. Arranging the two Equations (84) and (85) and using Equation (81), we obtain the yield surface in the(χ_p, χ_q) plane:

$$\chi_q = \chi_p M_g \frac{(1+c)}{c}\left(1-\left(\frac{p'}{p_g}\right)^c\right). \tag{86}$$

The strain rate vector is orthogonal to this yield surface, so that the normality rule holds in generalized stress space, as predicted by theory.

Therefore we can deduce that $\chi_p = p'$ and $\chi_q = q$, which presume that there is no back stress. The yield condition (86) becomes in the true stress space

$$q = pM_f \frac{(1+c)}{c}\left(1-\left(\frac{p'}{p_f}\right)^c\right) \tag{87}$$

obtained by use of Equation (81). This is the yield condition of the BSZ model (14).

Finally, the dissipation function (80) is always non-negative for the same reasons of Equation (62), the material parameters, M_g, p_g, c, are taken positive with the effective stress tensor p' and the deviatoric part of plastic strain rate increment coincides with the rate of plastic multiplier (74), $\dot{\varepsilon}_s^p = \dot{\lambda}$, so, for the Kuhn–Tucker conditions, $\dot{\lambda} \geqslant 0$, $f(\sigma, k) \leqslant 0$, $\dot{\lambda} f(\sigma, k) = 0$, the deviatoric part of plastic strain rate increment is always non-negative.

No back stresses are introduced and hence the Gibbs and Helmholtz free energies are the same as in the case of associative plasticity.

The BSZ model is hence derived from two potential functions and from a dissipative function taking into account of associative and non associative flow rule. The thermodynamic conditions are not violated because it has been demonstrated that the dissipation function is positive in both cases, so this assures the thermodynamic consistency of the model.

4. Comparison with Experimental Data

Extensive comparisons of the BSZ model with experimental data have been shown in Bolzon *et al.*(1996). Here we compare the model predictions with further available experimental data. The data used have been obtained from the extensive experimental investigation conducted by Geiser (1999). The soil used is a slime of the Sion region in Switzerland; composed by slime (72%), clay (8%) and sand (20%). First the saturated behaviour has been studied, and then, on the same paths, the suction influence.

The list of saturated normal consolidated specimens of Sion slime is reported in Table I. In the saturated specimens $s = 0$.

Figures 6 and 7 show the deviator behaviour of the saturated soil analysed in drained conditions. In the $(\varepsilon_1 - q)$ plane, the maximum deviator

Table I. Saturated normal consolidated specimens

Cell pressure $\sigma_3(kPa)$	s	$\Delta q/\Delta p'$	e_i	e_f
100	0	3	0.73	0.665
200	0	3	0.755	0.683
400	0	3	0.712	0.59
500	0	3	0.712	0.59
600	0	3	0.724	0.62
200	0	3	0.74	0.63
400	0	3	0.76	0.64

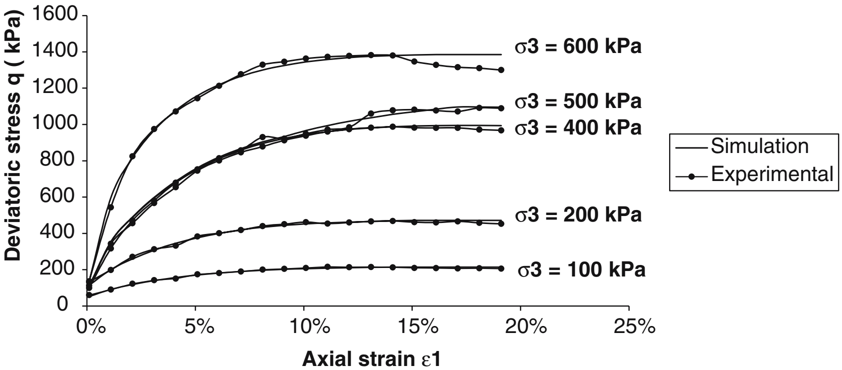

Figure 6. Triaxial test normal consolidated specimen in drained and fully saturated conditions.

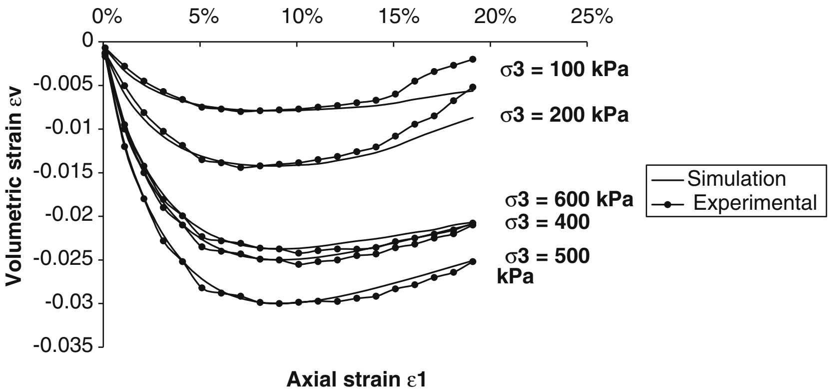

Figure 7. Triaxial test normal consolidated specimen in drained and fully saturated conditions.

stress increases regularly with the cell pressure σ_3. In the $(\varepsilon_1 - \varepsilon_v)$ plane, the behaviour is initially contracting and then weakly expanding, like a over-consolidated soil. The expansion is marked for the lower cell pressures σ_3.

Figures 8 and 9 show the behaviour of the unsaturated soil analysed in drained conditions with the same value of suction 100 kPa and different cell pressures σ_3. Figures 10 and 11 show the behaviour of the unsaturated soil with the same cell pressure σ_3 for different values of suction.

The list of unsaturated normal consolidated specimens of Sion slime is reported in Table II.

The behaviour of unsaturated specimens is similar to that analysed in saturated conditions, but the suction produces a type of weakly

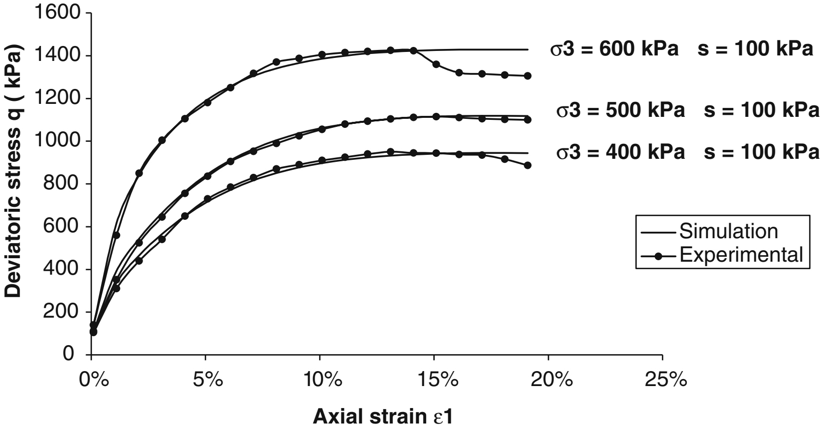

Figure 8. Triaxial test normal consolidated partially saturated specimen in drained conditions with suction 100 kPa.

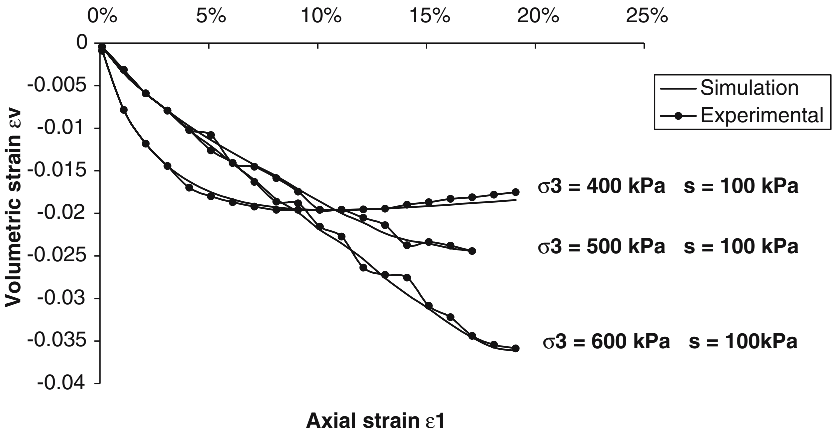

Figure 9. Triaxial test normal consolidated partially saturated specimen in drained conditions with suction 100 kPa.

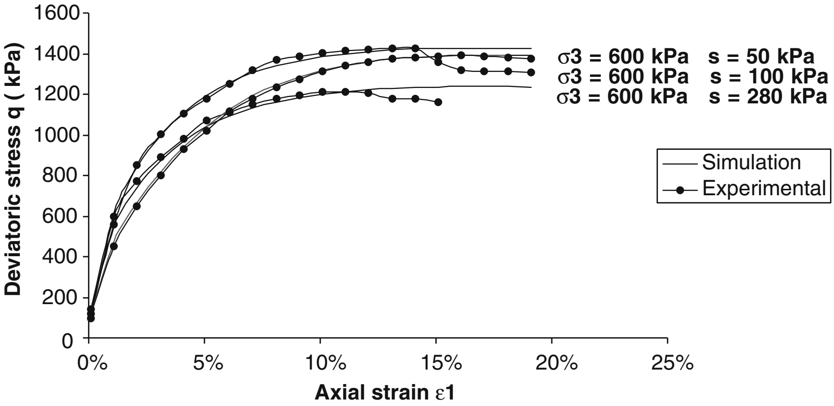

Figure 10. Triaxial test normal consolidated partially saturated specimen in drained conditions for different values of suction.

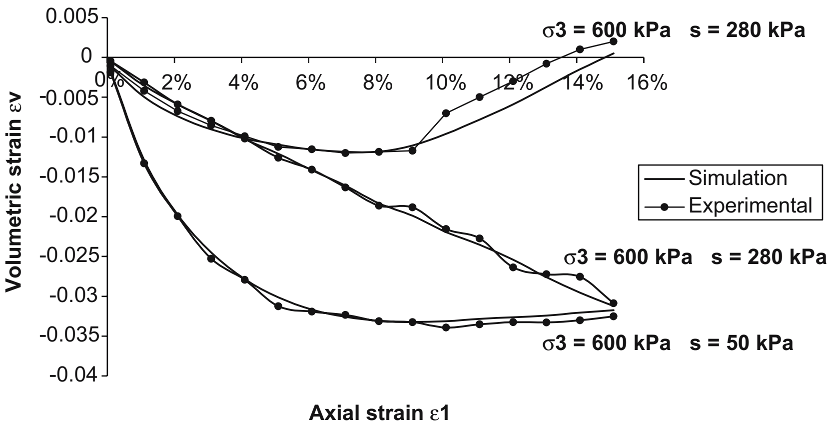

Figure 11. Triaxial test normal consolidated partially saturated specimen in drained conditions for different values of suction.

Table II. Unsaturated normal consolidated specimens

Cell pressure $\sigma_3(kPa)$	S	$\Delta q/\Delta p'$	e_i	e_f
400	100	3	0.802	0.66
500	100	3	0.712	0.63
600	50	3	0.755	0.62
600	100	3	0.788	0.64
600	280	3	0.712	0.67

overconsolidated behaviour; e.g. for the specimens at suction of 100 kPa, Figure 8, there is a peak and then a decrease of the deviator stress q until a last lower value. This behaviour is neglected in our model.

For the specimens at suction of 50 kPa the behaviour is very similar to that of saturated normal consolidated specimens (compare Tables I and II).

The parameters of the BSZ model to simulate the Sion slime are reported in the Table III and make reference to the parameters of Pastor *et al.* (1990). *Kev0* and Ges0 are the elastic constants k and G and H_{U0} is a constant in the plastic modulus for unloading.

The prediction of the model for drained boundary condition test is very close to the experimental data in saturated and also in partially saturated conditions.

The model can predict the following features of unsaturated soils:

- the increase of final shear strenght with suction, at constant p',
- the increase of preconsolidation pressure with suction,
- the decrease of mechanical compressibility with suction,
- the collapse.

But it has some limits:

- it cannot predict the fragile post – peak behaviour observed for the strong values of suction,
- in the prediction of volumetric strain, the model's tendency to dilation is close to the experimental data, but, sometimes, like in Figure 8 with $\sigma_3 = 100$ kPa , it is too marked.

The behaviour of the model in case of a wetting and drying cycle is investigated next.The following graphs (Figures 12 and 13) show the experimental data of Sharma (1998), reported in Wheeler *et al.* (2003), of a wetting and drying cycle on compacted betonite–kaolin performed under

Table III. Parameters of the model

Parameter	Value of parameter
Kev0	10,000 kPa
Ges0	20,000 kPa
M_f	1.3
M_g	1.3
β_0	1.5
β_1	0.1
c	0.45
H_{U0}	250

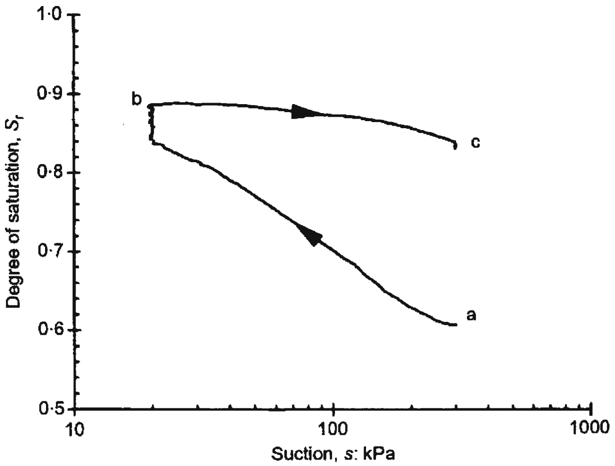

Figure 12. Wetting–drying cycle on compacted betonite–kaolin performed under isotropic stress-state in (s, S_w) plane. Redrawn with permission from Wheeler *et al.* (2003).

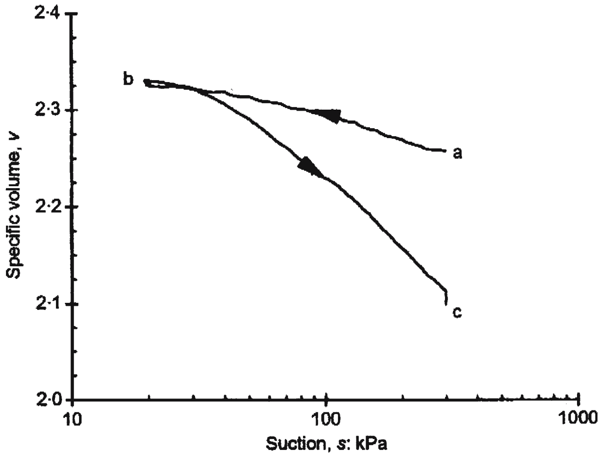

Figure 13. Wetting-drying cycle on compacted betonite–kaolin performed under isotropic stress-state in (s, v) plane. Redrawn with permission from Wheeler *et al.* (2003).

isotropic stress state. This author uses the net mean stress, i.e the difference between the total stress and the air pressure. The model predictions (Figures 14 and 15) follow the paths from suction $s = 200\,\text{kPa}$ to $s = 20\,\text{kPa}$ and back to $s = 200\,\text{kPa}$, at a constant mean net stress of 10 kPa. For comparison, the mean net stress, $\bar{p}$, is related to the effective mean stress as in Bolzon *et al.* (1996):

$$p' = \bar{p} + S_w s. \tag{88}$$

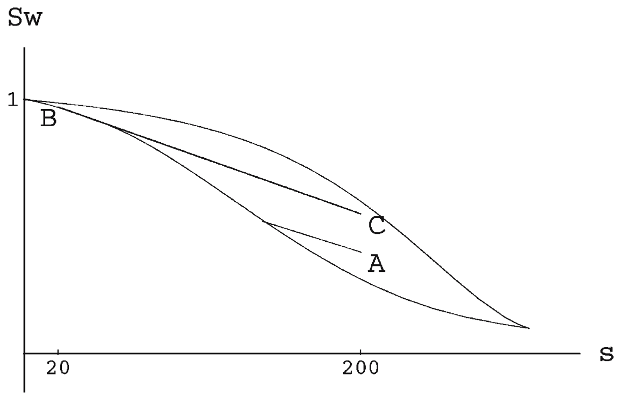

Figure 14. Wetting–drying cycle model prediction performed at a constant mean net stress of 10 kPa in (s, S_w) plane.

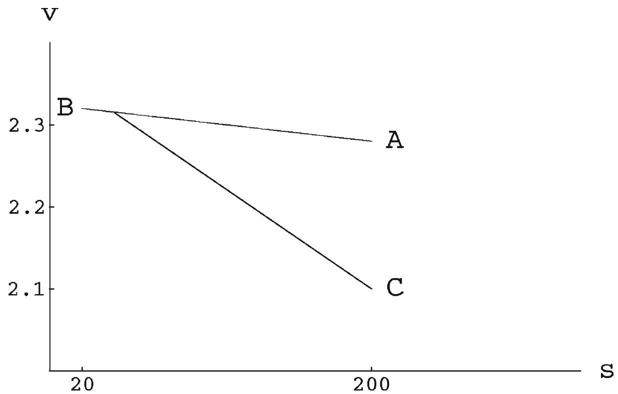

Figure 15. Wetting–drying cycle model prediction performed at a constant mean net stress of 10 kPa in (s, v) plane.

Wetting from point A to B (Figure 14) first follows the scanning curve with elastic increment of the degree of water saturation until this reaches the main wetting curve ψ_{SD}. The wetting then follows this elasto-plastic curve until point B with plastic increment of degree of water saturation. In drying from point B to C there is only elastic change in degree of saturation until the suction reaches the initial value; the difference in the degree of water saturation between point A and C is due to the hysteresis.

In Figure 15, where v indicates the specific volume (as in Figure 13), related to volumetric strain by

$$\varepsilon_v = \frac{v - v_0}{v_0}, \tag{89}$$

elastic volumetric strains are produced in the wetting path from point A to B and also in the initial response during drying from B to C. In the second part of the BC drying the path crosses the yield surface and plastic volumetric strains occur. The comparison of the model predictions (Figure 15) with the experimental data (Figure 13) shows that the model can successfully represent the elastic expansion during a wetting path and subsequently the irreversible compression during a drying path that does not exceed the maximum value of suction previously applied. The parameters used in the simulation are the same as those used by Wheeler *et al.* (2003) to simulate this problem: $\bar{\lambda} = 0.15$, $k = 0.02$, $v_0 = 2.2$ and the initial degree of water saturation $S_w = 0.65$. The value of parameter w, Equation (42), is assumed to be equal to 10.

5. Conclusions

An enhanced elasto-plastic model for partially saturated soil has been presented, starting from the motivation of the choice of the adopted effective stress tensor: the form of the tensor, initially proposed by Lewis and Schrefler, is thermodynamically consistent and consistent with the adopted strain measures. Further its use is straightforward and no particular condition is needed when passing from fully saturated to partially saturated conditions.

Then the constitutive model has been described which takes now into account also the hydraulic constitutive relationship, hydraulic hysteresis, and a new term of plastic strain introduced to account for irreversible deformation during cyclic drying and wetting until structural collapse.

A thermodynamic formulation of this model has been proposed, both for associative and non-associative plasticity, with a dissipation function and a de-coupled Helmholtz free energy to derive the constitutive equations and the collapse behaviour.

The model predictions of the behaviour of a Sion slime have been compared with available experimental data, with special attention on the deviatoric part. This comparison is satisfactory. The few shortcomings of the model have also been pointed out. At last it is shown that our model is able to simulate well the behaviour of compacted betonite–kaolin during a wetting and drying cycle, performed under isotropic stress state.

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